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On p-adic q-L-functions and sums of powers

Taekyun Kim *

Jangjeon Research Institute for Mathematical Science, 252-5 Hapcheon-Dong Hapcheon-Gun Kyungshang Nam-Do, 678-800, South Korea

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Abstract

We give an explicit p-adic expansion of $\sum_{j=1}^{*np} q^j/[j]^r$ as a power series in n which generalizes the formula of Andrews (Discrete Math. 204 (1999) 15). Indeed, this is a q-analogue result due to Washington (J. Number Theory 69 (1998) 50), corresponding to the case $q=1$. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$.

If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leqslant 1$. We use the notation:

$$[x] = [x:q] = \frac{1-q^x}{1-q}.$$

Note that when p is prime $[p]$ is an irreducible polynomial in $Q(q)$. Furthermore, this means that $Q(q)/[p]$ is a field and consequently rational functions $r(q)/s(q)$ are well defined modulo $[p]$ if $(r(q), s(q)) = 1$. Recently Andrews (see [1]) presented q-analogs of several classical binomial coefficient congruences due to Babbage, Wolstenholme and Glaisher. In [1], Andrews has proved the q-analogue of Wolstenholme' harmonic

* Present address: Institute of Science Education, Kongju National University, Kongju 314-701, South Korea.
E-mail address: tkim@kongju.ac.kr or taekyun64@hanmail.net (T. Kim).

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series congruence. Let $L_{p,q}(s,\chi)$ be the q-analogue of the p-adic L-function attached to a character χ (see [3]).

In this paper, I give new interesting results on the p-adic (q-L)-functions, a subject initiated by Neal Koblitz [7], in the beginning of the 1980's in which the author made some contributions, [3]. The main result of this paper, formula (37), is actually a q-analogue of a result of Washington [8], corresponding to the case $q=1$.

Deeba and Rodriguez (see [2]) have proved the following recurrence:

$$n(1-n^m)B_m=\sum_{k=0}^{m-1} n^k \binom{m}{k} B_k \sum_{j=1}^{n-1} j^{m-k}, \tag{1}$$

where B_k is the kth ordinary Bernoulli number. We give a q-analogue of formula (1) in Section 3.

2. Preliminaries

For $n\in\mathbb{N}=\{1,2,3,\ldots\}$, we define the q-Bernoulli numbers, $\beta_m(q^n)$ as (cf. [3])

$$\beta_m(q^n)=\int_{\mathbb{Z}_p} q^{-nx}[x:q^n]^m \,\mathrm{d}\mu_{q^n}(x), \tag{2}$$

where $\mu_{q^n}(x)=\mu_{q^n}(x+p^N\mathbb{Z}_p)=q^{nx}/[p^N:q^n]$, cf. [5].

Note that $\beta_m=\beta_m(q)$ are not the same as Carlitz's q-Bernoulli numbers, which are defined by

$$\beta_n^*=\beta_n^*(q)=\int_{\mathbb{Z}_p}[x]^n \,\mathrm{d}\mu_q(x) \text{ (see [3]).}$$

Thus we have, cf. [3]

$$\beta_0(q^n)=\frac{q^n-1}{n\log q} \quad (q^n\beta(q^n)+1)^k-\beta_k(q^n)=\begin{cases}1 & \text{if } k=1,\\ 0 & \text{if } k>1\end{cases}$$

with the usual convention about replacing $\beta^i(q^n)$ by $\beta_i(q^n)$.

Note that the $\lim_{q\to 1}\beta_k(q^n)=B_k$.

It is easy to see in [3,5] that

$$\beta_m(q^n)=\frac{1}{(q^n-1)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\frac{i}{[i:q^n]}.$$

Note that

$$\beta_m=\frac{1}{(q-1)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^{m-i}\frac{i}{[i]}. \tag{3}$$

The q-Bernoulli numbers β_m could be easily defined by the formulas in (3) as follows:

$$G_q(t) = \mathrm{e}^{t/(1-q)} \sum_{j=0}^{\infty} \frac{j}{[j]} (-1)^j \left(\frac{1}{1-q} \right)^j \frac{t^j}{j!}$$
$$= \sum_{j=0}^{\infty} \frac{\beta_j}{j!} t^j \quad \text{for } |t| < 1. \tag{4}$$

Thus we see that the q-Bernoulli numbers are the unique solutions of the following q-difference equation in the complex plane:

$$G_q(t) = \frac{q-1}{\log q} \mathrm{e}^{t/(1-q)} - t \sum_{n=0}^{\infty} q^n \mathrm{e}^{[n]t} \quad \text{for } |t| < 1, \tag{5}$$

where $q \in \mathbb{C}$ with $|q| < 1$.

For $r \in \mathbb{N}$, we define the q-Bernoulli polynomials $\beta_n(x, q^r)$ as

$$\beta_n(x, q^r) = \int_{\mathbb{Z}_p} q^{-rt} [x + t : q^r]^n \, \mathrm{d}\mu_{q^r}(t) \quad (\text{cf. [3]}). \tag{6}$$

Note that $\beta_n(x, q)$ is not the same as Carlitz's q-Bernoulli polynomials, which are defined by

$$\beta_n^*(x, q) = \int_{\mathbb{Z}_p} [x + t]^n \, \mathrm{d}\mu_q(t) \quad (\text{cf. [3]}).$$

It is easy to see that

$$\beta_n(x, q^r) = (q^{rx} \beta(q^r) + [x : q^r])^n = \sum_{j=0}^{n} \binom{n}{j} \beta_j(q^r) q^{rjx} [x : q^r]^{n-j}$$
$$= \frac{1}{(1-q^r)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{k}{[k : q^r]} q^{rkx} (-1)^k.$$

Note that

$$\beta_n(x, q) = (q^x \beta + [x])^n = \sum_{j=0}^{n} \binom{n}{j} \beta_j q^{jx} [x]^{n-j}$$
$$= \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{k}{[k]} q^{kx} (-1)^k. \tag{7}$$

Then (7) can be used to define the polynomials as follows:

$$\sum_{n=0}^{\infty} \beta_n(x, q) \frac{t^n}{n!} = G_q(x, t) = \mathrm{e}^{t/(1-q)} \sum_{j=0}^{\infty} \frac{j}{[j]} (-1)^j \left(\frac{1}{1-q} \right)^j q^{jx} \frac{t^j}{j!} \quad \text{for } |t| < 1. \tag{8}$$

By (8), we obtain the following q-difference equation in the complex plane:

$$G_q(x, t) = \frac{q-1}{\log q} \mathrm{e}^{t/(1-q)} - t \sum_{n=0}^{\infty} q^{n+x} \mathrm{e}^{[n+x]t} \quad \text{for } |t| < 1, \tag{9}$$

where $q \in \mathbb{C}$ with $|q| < 1$.

It is shown in [3,5] that

$$[d]^{k-1}\sum_{i=0}^{d-1}\beta_k\left(\frac{x+i}{d},q^d\right)=\beta_k(x,q), \tag{10}$$

where d,k are positive integers.

3. On identities of q-Bernoulli numbers and q-analogue of Dirichlet L-series (in complex)

In this section, we assume $q\in\mathbb{C}$ with $|q|<1$. Now, we prove the q-analogue of (1) by using (10). If we put $x=0$ in (10), then we obtain

$$[n]\beta_m=\sum_{k=0}^{m}\binom{m}{k}\beta_k(q^n)[n]^k\sum_{j=0}^{n-1}q^{jk}[j]^{m-k}. \tag{11}$$

It is easy to see (11) using (7). By (11), we see

$$[n]\beta_m-\beta_m(q^n)[n]^m\frac{[mn]}{[m]}=\sum_{k=0}^{m-1}\binom{m}{k}[n]^k\beta_k(q^n)\sum_{j=1}^{n-1}q^{jk}[j]^{m-k}. \tag{12}$$

Define the operation $*$ on $f(q)$ as follows:

$$[n](1-[n]^m)*f(q)=[n]f(q)-[n]^m\frac{[mn]}{[m]}f(q^n).$$

Thus (12) can be written using $*$ as

$$[n](1-[n]^m)*\beta_m(q)=\sum_{k=0}^{m-1}\binom{m}{k}[n]^k\beta_k(q^n)\sum_{j=1}^{n-1}q^{jk}[j]^{m-k}. \tag{13}$$

Note that (13) is the q-analogue of (1). It was well known that, for positive integers s and n,

$$\sum_{l=0}^{n-1}l^{s-1}=\frac{1}{s}\sum_{j=0}^{s-1}\binom{s}{j}B_jn^{s-j}. \tag{14}$$

Now, we give the q-analogue of (14) which is used later. It is easy to see that

$$-\sum_{l=0}^{\infty}q^{l+n}\mathrm{e}^{[n+l]t}+\sum_{l=0}^{\infty}q^l\mathrm{e}^{[l]t}=\sum_{m=1}^{\infty}\left(m\sum_{l=0}^{n-1}q^l[l]^{m-1}\right)\frac{t^{m-1}}{m!}, \tag{15}$$

where $q\in\mathbb{C}$ with $|q|<1$.

By (4), (8), (15), we obtain the following:

$$\beta_m(n,q)-\beta_m=m\sum_{l=0}^{n-1}q^l[l]^{m-1}.$$

Hence, we can obtain the q-analogue of (14) as follows:

$$\sum_{l=0}^{n-1}q^l[l]^{m-1}=\frac{1}{m}\sum_{l=0}^{m-1}\binom{m}{l}q^{nl}\beta_l[n]^{m-l}+\frac{1}{m}(q^{mn}-1)\beta_m. \tag{16}$$

Differentiating both sides with respect to t and comparing coefficients in (5), we have (cf. [4]).

$$\beta_k(x,q) = -\frac{(1-q)^{1-k}}{\log q} - k\sum_{n=0}^{\infty} q^{n+x}[n+x]^{k-1}. \tag{17}$$

By (17), we define the q-analogue of the Hurwitz zeta function: For $s \in \mathbb{C}$,

$$\zeta_q(s,a) = \frac{(1-q)^s}{(1-s)\log q} + \sum_{n=0}^{\infty} \frac{q^{n+a}}{[n+a]^s}, \tag{18}$$

where a is a real number with $0 < a \leqslant 1$, and $q \in \mathbb{C}$ with $|q| < 1$.

Note that $\zeta_q(1-k,a) = -\beta_k(a,q)/k$, where k is any positive integer (see [3,4]).

Let

$$\begin{aligned} J_q(s,a,F) &= \sum_{\substack{m \equiv a (\bmod F) \\ m > 0}} \frac{q^m}{[m]^s} + \frac{1}{F}\frac{(1-q)^s}{(1-s)\log q} \\ &= \sum_{n=0}^{\infty} \frac{q^{a+nF}}{[a+nF]^s} + \frac{1}{F}\frac{(1-q)^s}{(1-s)\log q} \\ &= [F]^{-s}\zeta_{q^F}\left(s, \frac{a}{F}\right), \end{aligned} \tag{19}$$

where a and F are positive integers with $0 < a < F$. Then

$$J_q(1-n,a,F) = -\frac{[F]^{n-1}\beta_n(a/F,q^F)}{n}, \quad n \geqslant 1, \tag{20}$$

and J_q has a simple pole at $s=1$ with residue $(q-1)/\log q\, 1/F$.

Let χ be the Dirichlet character with conductor F. Then we define the q-analogue of the Dirichlet L-series as follows: For $s \in \mathbb{C}$,

$$L_q(s,\chi) = \sum_{a=1}^{F} \chi(a) J_q(s,a,F).$$

The functions $J_q(s,a,F)$ will be called the q-analogue of the partial zeta functions. Now, we define the generalized q-Bernoulli numbers with χ as follows:

$$\beta_{k,\chi} = \beta_{k,\chi}(q) = [F]^{k-1}\sum_{a=1}^{F} \chi(a)\beta_k\left(\frac{a}{F} : q^F\right).$$

Note that

$$L_q(1-k,\chi) = -\frac{\beta_{k,\chi}}{k} \quad \text{for } k \geqslant 1.$$

Remark. The numbers $\beta_{k,\chi}$ are not the same as the generalized Carlitz's q-Bernoulli numbers with χ which are defined in [3].

The values of $L_q(s,\chi)$ at negative integers are algebraic, hence may be regarded as lying in an extension of $\mathbb{Q}_p$. We therefore look for a p-adic function which agrees with $L_q(s,\chi)$ at the negative integers in Section 4.

4. p-Adic q-L-functions

Let p be an odd prime and let $L_{p,q}(s,\chi)$ be the q-analogue of the p-adic L-function attached to a character χ (see [3]). We define $\langle x\rangle = \langle x : q\rangle = [x]/\omega(x)$, where $\omega(x)$ is the Teichmüller character. When F is a multiple of p and $(a,p)=1$, we define a p-adic analogue of (19) as

$$J_{p,q}(s,a,F) = \frac{1}{s-1}\frac{1}{[F]}\langle a\rangle^{1-s}\sum_{j=0}^{\infty}\binom{1-s}{j} q^{ja}\left(\frac{[F]}{[a]}\right)^{j}\beta_j(q^F) \tag{21}$$

for $s \in \mathbb{Z}_p$. It is easy to see in (7), (20), (21) that

$$\begin{aligned} J_{p,q}(1-n,a,F) &= -\frac{1}{n}\frac{1}{[F]}\langle a\rangle^{n}\sum_{j=0}^{n}\binom{n}{j}\beta_j(q^F)q^{ja}\left(\frac{[F]}{[a]}\right)^{j} \\ &= -\frac{1}{n}[F]^{n-1}\omega^{-n}(a)\sum_{j=0}^{n}\binom{n}{j}\beta_j(q^F)q^{F(ja/F)}[a:q^F]^{n-j} \\ &= -\frac{1}{n}[F]^{n-1}\omega^{-n}(a)\beta_n\left(\frac{a}{F},q^F\right) = \omega^{-n}(a)J_q(1-n,a,F), \end{aligned} \tag{22}$$

for all positive integers n and it has a simple pole at $s=1$ with residue $(q-1)/\log q\, 1/F$.

It is easy to see from [3], (21), (20) that

$$L_{p,q}(s,\chi) = \sum_{\substack{a=1 \\ p\nmid a}}^{F}\chi(a)J_{p,q}(s,a,F). \tag{23}$$

For $f \in \mathbb{N}$, let χ be the Dirichlet character with conductor f. The generalized q-Bernoulli numbers with character χ, which were defined in Section 3, is associated with a p-adic q-integral as follows:

$$\beta_{k,\chi}(q) = \int_{\mathbb{Z}_p}\chi(x)q^{-x}[x]^k\,\mathrm{d}\mu_q(x) \quad \text{(cf. [3])}, \tag{24}$$

where k is a positive integer.

We see in (24) that

$$\beta_{k,\chi}(q) = [f]^{k-1}\sum_{a=0}^{f-1}\chi(a)\beta_k\left(\frac{a}{f},q^f\right). \tag{25}$$

By (22), (23), (25), if $n \geqslant 1$ then we have

$$\begin{aligned} L_{p,q}(1-n,\chi) &= \sum_{\substack{a=1 \\ p\nmid a}}^{F}\chi(a)J_{p,q}(1-n,a,F) \\ &= -\frac{1}{n}(\beta_{n,\chi\omega^{-n}}(q) - [p]^{n-1}\chi\omega^{-n}(p)\beta_{n,\chi\omega^{-n}}(q^p)). \end{aligned} \tag{26}$$

In fact, we have the formula

$$L_{p,q}(s,\chi)=\frac{1}{[F]}\frac{1}{s-1}\sum_{\substack{a=1\\ p\nmid a}}^{F}\chi(a)\langle a\rangle^{1-s}\sum_{j=0}^{\infty}\binom{1-s}{j}\beta_j(q^F)q^{ja}\left(\frac{[F]}{[a]}\right)^j, \tag{27}$$

for $s\in\mathbb{Z}_p$.

This is a p-adic analytic function (except possibly at $s=1$) and has the following properties for $\chi=\omega^t$:

$$L_{p,q}(1-k,\omega^t)=-\frac{1}{k}(\beta_k(q)-[p]^{k-1}\beta_k(q^p)), \tag{28}$$

where $1\leqslant k\equiv t \pmod{p-1}$,

$$L_{p,q}(s,\omega^t)\in\mathbb{Z}_p \quad \text{for all } s\in\mathbb{Z}_p \text{ when } t\not\equiv 0 \pmod{p-1}. \tag{29}$$

If $t\not\equiv 0 \pmod{p-1}$, then

$$L_{p,q}(s_1,\omega^t)\equiv L_{p,q}(s_2,\omega^t) \pmod{p} \quad \text{for all } s_1,s_2\in\mathbb{Z}_p. \tag{30}$$

$L_{p,q}(s,1)$ has a simple pole at $s=1$ with residue $(p-1)/[p](q^F-1)/\log q^F$,

$$L_{p,q}(k,\omega^t)\equiv L_{p,q}(k+p,\omega^t) \pmod{p}. \tag{31}$$

The proofs of (28)–(31) can be found in [3]. It is easy to see that

$$\frac{1}{r+k-1}\binom{-r}{k}\binom{1-r-k}{j}=\frac{-1}{j+k}\binom{-r}{k+j-1}\binom{k+j}{j} \tag{32}$$

for all positive integers r,j,k with $j,k\geqslant 0$, $j+k>0$, and $r\neq 1-k$.

Let

$$B^{(n)}_{p,q}(s,a,F)=\sum_{l=1}^{\infty}\tfrac{1}{l}[a]^{-s}q^{la}[Fln]\binom{-r}{l-1}\left(\frac{[F]}{[a]}\right)^{l-1}\beta_l(q^F),$$

$$J^{(n)}_{p,q}(s,a,F)=\frac{1}{s-1}\frac{1}{[F]}\langle a\rangle^{1-s}\sum_{j=0}^{\infty}\binom{1-s}{j}[nFj]\beta_j(q^F)q^{ja}\left(\frac{[F]}{[a]}\right)^j \tag{33}$$

for all $s\in\mathbb{Z}_p$.

By (16), (21), (22), (32), (33), we can obtain the following, for $r\geqslant 1$:

$$\begin{aligned}&\sum_{k=1}^{\infty}\binom{-r}{k}\omega^{1-r-k}(a)[Fn]^kq^{ak}J_{p,q}(r+k,a,F)\\ &\quad+(q-1)\left\{\sum_{k=1}^{\infty}\binom{-r}{k}\omega^{1-r-k}(a)[Fn]^kq^{ak}J^{(n)}_{p,q}(r+k,a,F)+B^{(n)}_{p,q}(r,a,F)\right\}\\ &\quad=-\sum_{l=0}^{n-1}\frac{q^{Fl+a}}{[a+Fl]^r}.\end{aligned} \tag{34}$$

For $F=p, r\in\mathbb{N}$, we see that

$$\sum_{a=1}^{p-1}\sum_{l=0}^{n-1}\frac{q^{a+lp}}{[a+pl]^r}=\sideset{}{^*}\sum_{j=1}^{np}\frac{q^j}{[j]^r}, \tag{35}$$

where $*$ means to take sum over the rational integers prime to p in the given range. We set

$$T_{p,q}(r+k,\omega^{1-k-r}) = \sum_{a=1}^{p-1}\omega^{1-k-r}(a)\{[ak]J_{p,q}(r+k,a,p) + q^{ak}J_{p,q}^{(n)}(r+k,a,p)\} \tag{36}$$

for all $r \geqslant 1, n \geqslant 1$. By (23), (27), (34),(35), (36), we obtain the following:

Theorem. *Let p be an odd prime and let $n \geqslant 1$, and $r \geqslant 1$ be integers. Then*

$$\begin{aligned}\sum_{j=1}^{\overset{np}{*}} \frac{q^j}{[j]^r} = &-\sum_{k=1}^{\infty}\binom{-r}{k}[pn]^k L_{p,q}(r+k,\omega^{1-k-r}) \\ &-(q-1)\left\{\sum_{k=1}^{\infty}\binom{-r}{k}[pn]^k T_{p,q}(r+k,\omega^{1-k-r})\right. \\ &\left.+\sum_{a=1}^{p-1}B_{p,q}^{(n)}(r,a,F)\right\}.\end{aligned} \tag{37}$$

Note, this result at $q=1$ is explicitly given in [8] by more or less the same method.

Remark. (1) For $r=n=1$ in (37), we can obtain the following congruence:

$$\sum_{j=1}^{p-1}\frac{q^j}{[j]} \equiv \frac{(p-1)}{2}(q-1) \pmod{[p]}. \tag{38}$$

Formula (38) is the result of Andrews (see [1]).

(2) For $q=1$ in (37), we have the following formula:

$$\sum_{j=1}^{\overset{np}{*}}\frac{1}{j^r} = -\sum_{k=1}^{\infty}\binom{-r}{k}(pn)^k L_p(r+k,\omega^{1-k-r}) \quad \text{(see [8])}, \tag{39}$$

where $L_p(s,\omega^t)$ is the p-adic L-function.

The result of Washington from [8] was apparently proved by Barsky. See the Math. Review of [8].

In the case of $r=1$, (39) is related to the formula of Wolstenholme (see [1]).

5. Uncited reference

[6]

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